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DEPARTMENT OF CIVIL ENGINEERING

FORCED MOTIONS OF TIMOSHENKO BEAMS

by

G. HERRMANN

Office of Naval Research Project NR-064-388
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ABSTRACT

Timoshenko's theory of flexural motions in an elastic beam takes into account both rotatory inertia and transverse shear deformation and, accordingly, contains two dependent variables instead of the one transverse displacement of classical theory of flexure. For the case of forced motions, the solution involves complications not usually encountered. The difficulties may be surmounted in several ways, one of which is presented in this paper. The method described makes use of the property of orthogonality of the principal modes of free vibration and uses the procedure of R. D. Mindlin and L. E. Goodman in dealing with time-dependent boundary conditions. Thus, the most general problem of forced motion is reduced to a free vibration problem and a quadrature.

FORCED MOTIONS OF TIMOSHENKO BEAMS

Introduction

Timoshenko's theory of flexural motions of elastic beams, in contrast to the elementary Bernoulli-Euler theory, allows for corrections for shear deformation and rotatory inertia. Though derived more than 30 years ago (1)[1], and attracting the attention of many workers in the field[2], no complete solution of the most general boundary value problem of a finite beam with time-dependent boundary conditions of any admissible combination, acted upon by time-dependent normal loads and bending moments with specified arbitrary initial conditions, seems to have been developed to date.

A major difficulty in solving boundary value problems governed by Timoshenko-type (4), (5) equations, consists in finding a proper combination of two (or more) "displacement" components, which would exhibit the property of orthogonality. This property is essential in developing series solutions by the classical (D. Bernoulli) method of separation of variables. In a recent paper (6) it was shown, by means of the example of a Timoshenko-type theory of longitudinal motions of rods, how this difficulty may be overcome by making use of Lagrange's equations of motion.

An additional difficulty, typical of any vibration problem with initial and boundary conditions, arises when one or more boundary conditions are time-dependent. In (6) it was demonstrated, how this complication may be removed by solving, in certain cases, certain static problems.

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1. Numbers in parentheses refer to the Bibliography at the end of the paper.

2. An account of the problem is given in the recent papers (2) and (3) and in the bibliographies to these papers.

In contrast to the procedure of (6), and as a counterpart of it, we attack here the general forced vibration problem directly, without recourse to Lagrange's equations. Thereby, the time-dependency of the boundary conditions is dealt with, by making use of the idea of (7).

Thus, the most general boundary value problem of a finite Timoshenko beam, will be reduced to the solution of the corresponding free vibration problem and an integration.

Statement of Problem

Timoshenko's theory of flexural motions of elastic beams is contained[3] in the equations of motion

$$\begin{aligned} k'(y''-\psi')G/\rho + q/\rho A &= \ddot{y} \\ E\psi''/\rho + k'(y'-\psi)G/\rho r^2 + M_s/\rho A &= \ddot{\psi} \end{aligned} \qquad [1]$$

the stress-displacement relations

$$\begin{aligned} M &= -EI\psi' \\ Q &= k'(y'-\psi)AG \end{aligned} \qquad [2]$$

the boundary conditions

---

3. For derivation of those equations, consult, for example, (8). M_s is the net bending moment, per unit of length, along the beam, due to the application of shear tractions on the cylindrical surface of the beam. This moment is usually omitted, and is retained in the present paper for the sake of completeness.

1. At each point along the length of the beam, one member of each of the products $\dot{y}q$ and $\dot{\psi}M_s$ must be specified.

2. At each end of the bar, one member of each of the products $\dot{y}Q$ and $\dot{\psi}M$ must be specified, [3]

and the initial conditions

The initial displacements and velocities must be specified. [4]

In [1] to [4] the symbols have the following meanings:

E	Young's modulus
G	shear modulus
k'	shear deflection coefficient
ρ	mass density
y	transverse displacement of a cross-section
ψ	angle of rotation of a cross-section
A	area of the cross-section
I	area moment of inertia of the cross-section
r	cross-sectional radius of gyration, $r^2 = I/A$
q	net transverse load, per unit of length
M_s	net (bending) moment, due to applied axial shears
M	bending moment
Q	transverse shear

Primes indicate differentiation with respect to x, the coordinate along the centroidal axis of the beam, and dots indicate differentiation with respect to time.

As a convenience in specifying the boundary conditions, the following notation will be employed:

$$B_1 = y \text{ or } Q \quad \text{at} \quad x = 0$$
$$B_2 = \psi \text{ or } M \quad \text{at} \quad x = 0$$
$$B_3 = y \text{ or } Q \quad \text{at} \quad x = \ell$$
$$B_4 = \psi \text{ or } M \quad \text{at} \quad x = \ell$$

where ℓ is the length of the beam.

Accordingly, the boundary conditions [3] become

$$B_i = f_i(t) \qquad i = 1, 2, 3, 4 \qquad [3']$$

in which the four functions $f_i(t)$ are prescribed.

The initial conditions [4] are specified by four arbitrary functions

$$\begin{aligned} y(x,0) &= y_0(x) \\ \psi(x,0) &= \psi_0(x) \\ \dot{y}(x,0) &= \dot{y}_0(x) \\ \dot{\psi}(x,0) &= \dot{\psi}_0(x) \end{aligned} \qquad [4']$$

Free Vibrations

As a preliminary to the study of forced vibrations, we shall be concerned with the problem of free vibration, specified by the homogeneous differential equations of motion

$$\begin{aligned} k'(y'' - \psi')G/\rho &= \ddot{y} \\ E\psi''/\rho + k'(y' - \psi)G/\rho r^2 &= \ddot{\psi} \end{aligned} \qquad [5]$$

and homogeneous boundary conditions

$$B_i = 0 \qquad i = 1, 2, 3, 4 \qquad [6]$$

Considering solutions in the form

$$y(x,t) = Y(x) \sin \omega t$$
$$\psi(x,t) = \Psi(x) \sin \omega t \qquad [7]$$

we may show that the equations of motion [5], together with the boundary conditions [6], are satisfied for an infinite set of discrete frequencies ω_n, each of which corresponds to a mode shape given by functions $Y_n(x)$ and $\Psi_n(x)$. These modes are determined, except for a multiplying factor, common to both functions, from the equations

$$G k' (Y_n'' - \Psi_n') = -\rho \omega_n^2 Y_n$$
$$E \Psi_n'' + k' (Y_n' - \Psi_n) G/r^2 = -\rho \omega_n^2 \Psi_n \qquad [8]$$

or, because of [2], from the equations

$$Q_n' = -\rho A \omega_n^2 Y_n$$
$$-M_n' + Q_n = -\rho I \omega_n^2 \Psi_n \qquad [9]$$

We proceed to formulate the property of orthogonality of the principal modes of free vibration, using Rayleigh's method (9).

We multiply the first of [9] by Y_m and the second of [9] by Ψ_m, integrate over the length of the beam and obtain, after integration by parts,

$$-\rho A \omega_n^2 \int_0^\ell Y_n Y_m \, dx = \int_0^\ell Q_n' Y_m \, dx = Q_n Y_m \Big]_0^\ell - \int_0^\ell Q_n Y_m' \, dx$$
$$-\rho I \omega_n^2 \int_0^\ell \Psi_n \Psi_m \, dx = \int_0^\ell (-M_n' + Q_n) \Psi_m \, dx = \qquad [10]$$
$$- M_n \Psi_m \Big]_0^\ell + \int_0^\ell (M_n \Psi_m' + Q_n \Psi_m) \, dx$$

Interchanging the subscripts m and n , a similar set of equations is obtained. Adding the two equations of each set and subtracting the resulting first from the resulting second relationship, we find

$$\rho(\omega_n^2-\omega_m^2)\int_0^\ell (A Y_n Y_m + I \Psi_n \Psi_m)\, dx =$$

$$-Q_n Y_m\Big]_0^\ell + Q_m Y_n\Big]_0^\ell + M_n \Psi_m\Big]_0^\ell - M_m \Psi_n\Big]_0^\ell \qquad [11]$$

$$-\int_0^\ell (M_n \Psi_m' - M_m \Psi_n')\, dx + \int_0^\ell [Q_n(Y_m' - \Psi_m) - Q_m(Y_n' - \Psi_n)]\, dx$$

The integrated terms in [11] vanish because of [6] and the two integrands in [11] vanish because of [2]. Thus, we obtain the formulation of the property of orthogonality of the principal modes of free vibration

$$\int_0^\ell (Y_n Y_m + r^2 \Psi_n \Psi_m)\, dx = 0\,, \qquad \omega_n \neq \omega_m \qquad [12]$$

Forced Vibrations

The solution of the general forced vibration problem, specified by [1], [3'] and [4'], will be sought in the form

$$y(x,t) = \sum_{i=1}^{4} g_{iy}(x) f_i(t) + s_y(x,t)$$
$$\qquad [13]$$
$$\Psi(x,t) = \sum_{i=1}^{4} g_{i\Psi}(x) f_i(t) + s_\Psi(x,t)$$

The eight functions g_{iy} , $g_{i\psi}$ correspond to the four functions g_i of (7). They are determined in such a way as to make the boundary conditions on the functions s_y and s_ψ homogeneous. This can be done for all possible combinations of boundary conditions, taking g_{iy} and $g_{i\psi}$ to be polynomials and choosing the coefficients accordingly. It is necessary only to compute those of the g_{iy} , $g_{i\psi}$ for which the corresponding $f_i(t)$ do not vanish.

If, as a first example, the boundary conditions are specified by

$$
\begin{aligned}
y(0) &= f_1(t) \\
M(0) &= f_2(t) \\
Q(\ell) &= f_3(t) \\
\psi(\ell) &= f_4(t)
\end{aligned}
$$

the functions g_{iy} , $g_{i\psi}$, which make the boundary conditions on s_y , s_ψ homogeneous, are calculated to be

$$
\begin{aligned}
g_{1y}(x) &= 1 \\
g_{1\psi}(x) &= 0 \\
g_{2y}(x) &= 0 \\
g_{2\psi}(x) &= (\ell - x)/EI \\
g_{3y}(x) &= x/Ak'G \\
g_{3\psi}(x) &= 0 \\
g_{4y}(x) &= x/\ell \\
g_{4\psi}(x) &= 1
\end{aligned}
$$

If, in the exceptional case, at both ends of the beam time-dependent moments and shears are specified by

$$Q(o) = f_1(t)$$

$$M(o) = f_2(t)$$

$$Q(e) = f_3(t)$$

$$M(e) = f_4(t)$$

the functions g_{iy}, $g_{i\psi}$ are found to be

$$g_{1y}(x) = -x^2/2eAk'G$$

$$g_{1\psi}(x) = -1/Ak'G$$

$$g_{2y}(x) = -x^2/4EI$$

$$g_{2\psi}(x) = x^2/2eEI - x/EI$$

$$g_{3y}(x) = 0$$

$$g_{3\psi}(x) = 2x^3/Ak'Ge^3 - 3x^2/Ak'Ge^2$$

$$g_{4y}(x) = 0$$

$$g_{4\psi}(x) = -x^3/EIe^2 + x^3/EIe$$

The remaining problem consists in finding $s_y(x,t)$ and $s_\psi(x,t)$ from the differential equations

$$\frac{k'G}{\rho}\left(\sum_{i=1}^{4} g''_{iy} f_i - \sum_{i=1}^{4} g'_{i\psi} f_i\right) + \frac{k'G}{\rho}(s''_y - s'_\psi) + q/\rho A$$
$$= \sum_{i=1}^{4} g_{iy} \ddot{f}_i + \ddot{s}_y$$

[14]

$$\frac{E}{\rho}\sum_{i=1}^{4} g''_{i\psi} f_i + \frac{E}{\rho} s''_\psi + \frac{k'G}{\rho r^2}\left(\sum_{i=1}^{4} g'_{iy} f_i - \sum_{i=1}^{4} g_{i\psi} f_i\right) + \frac{k'G}{\rho r^2}(s'_y - s_\psi) + M_s/\rho I$$
$$= \sum_{i=1}^{4} g_{i\psi} \ddot{f}_i + \ddot{s}_\psi$$

the boundary conditions

$$B_i = 0 \qquad i = 1, 2, 3, 4 \tag{15}$$

and the initial conditions

$$s_y(x,0) = y_0(x) - \sum_{i=1}^{4} g_{iy}(x) f_i(0)$$
$$s_\psi(x,0) = \psi_0(x) - \sum_{i=1}^{4} g_{i\psi}(x) f_i(0)$$
$$\dot{s}_y(x,0) = \dot{y}_0(x) - \sum_{i=1}^{4} g_{iy}(x) \dot{f}_i(0) \tag{16}$$
$$\dot{s}_\psi(x,0) = \dot{\psi}_0(x) - \sum_{i=1}^{4} g_{i\psi}(x) \dot{f}_i(0)$$

We seek a solution in the form

$$s_y(x,t) = \sum_{n=1}^{\infty} Y_n(x)\, T_n(t)$$
$$s_\psi(x,t) = \sum_{n=1}^{\infty} \Psi_n(x)\, T_n(t) \tag{17}$$

assuming[4] that the principal modes form a complete set.

---

4. This assumption could be justified, by generalizing the proofs of completeness given by Courant and Hilbert (10) and by Kemble (11) to the case of two dependent variables.

To find $T_n(t)$ we expand first

$$g_{iy} = \sum_{n=1}^{\infty} G_{in} Y_n$$
$$g_{i\psi} = \sum_{n=1}^{\infty} G_{in} \Psi_n \qquad [18]$$

Multiplication of the first of [18] by Y_m, of the second of [18] by $r^2 \Psi_m$, addition and integration over the length of the beam results, in view of [12], in

$$G_{in} = \frac{\int_0^{\ell} (g_{iy} Y_n + r^2 g_{i\psi} \Psi_n) \, dx}{\int_0^{\ell} (Y_n^2 + r^2 \Psi_n^2) \, dx} \qquad [19]$$

It may be observed that it is sufficient for the coefficients G_{in}, in the first and second of [18], to be the same.

We expand further

$$\frac{k'G}{\rho} (g_{iy}'' - g_{i\psi}') = \sum_{n=1}^{\infty} G_{in}^* Y_n$$
$$\frac{E}{\rho} g_{i\psi}'' + \frac{k'G}{\rho r^2} (g_{iy}' - g_{i\psi}) = \sum_{n=1}^{\infty} G_{in}^* \Psi_n \qquad [20]$$

Operations similar to those on [18] and again with the application of [12] yield

$$G_{in}^* = \frac{\int_0^{\ell} \left\{ \frac{k'G}{\rho} (g_{iy}'' - g_{i\psi}') Y_n + \left[\frac{E}{\rho} g_{i\psi}'' + \frac{k'G}{\rho r^2} (g_{iy}' - g_{i\psi}) \right] r^2 \Psi_n \right\} dx}{\int_0^{\ell} (Y_n^2 + r^2 \Psi_n^2) \, dx} \qquad [21]$$

Also we expand the applied time-dependent loads and moments

$$q/\rho A = \sum_{n=1}^{\infty} Q_n(t) Y_n(x)$$
$$M_s/\rho I = \sum_{n=1}^{\infty} Q_n(t) \Psi_n(x) \qquad [22]$$

In the same manner as in the expansions [18] and [20] we obtain the (time-dependent) coefficient in [22]

$$Q_n = \frac{\int_0^{\ell}\left(\frac{q}{\rho A}Y_n + \frac{M_s r^2}{\rho I}\Psi_n\right)dx}{\int_0^{\ell}\left(Y_n^2 + r^2\Psi_n^2\right)dx} \qquad [23]$$

Thus, each term in the equations [14] is an infinite series. Equating the n-th terms we obtain

$$\begin{aligned} \sum_{i=1}^{4} G_{in}^{*} Y_n f_i - \omega_n^2 Y_n T_n + Q_n Y_n &= \sum_{i=1}^{4} G_{in} Y_n \ddot{f}_i + \ddot{T}_n Y_n \\ \sum_{i=1}^{4} G_{in}^{*} \Psi_n f_i - \omega_n^2 \Psi_n T_n + Q_n \Psi_n &= \sum_{i=1}^{4} G_{in} \Psi_n \ddot{f}_i + \ddot{T}_n \Psi_n \end{aligned} \qquad [24]$$

where use has been made of [9].

Dividing the first equation of [24] by Y_n and the second by Ψ_n, there result two identical equations on T_n

$$\ddot{T}_n + \omega_n^2 T_n = -\sum_{i=1}^{4} G_{in}\ddot{f}_i + \sum_{i=1}^{4} G_{in}^{*} f_i + Q_n = P_n(t) \quad \text{say} \qquad [25]$$

the solution being

$$T_n = A_n \cos\omega_n t + B_n \sin\omega_n t + \frac{1}{\omega_n}\int_0^t P_n(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad [26]$$

The constants of integration A_n and B_n are determined from the initial conditions [16].

Expanding the initial displacements in a series of principal modes

$$\begin{aligned} y_0(x) &= \sum_{n=1}^{\infty} C_n Y_n \\ \psi_0(x) &= \sum_{n=1}^{\infty} C_n \Psi_n \end{aligned} \qquad [27]$$

as well as the initial velocities

$$\dot{y}_o(x) = \sum_{n=1}^{\infty} D_n Y_n \qquad [28]$$
$$\dot{\psi}_o(x) = \sum_{n=1}^{\infty} D_n \Psi_n$$

the expressions for the coefficients C_n and D_n are determined by making use of [12]

$$C_n = \frac{\int_0^{\ell} (y_o Y_n + r^2 \psi_o \Psi_n)\, dx}{\int_0^{\ell} (Y_n^2 + r^2 \Psi_n^2)\, dx} \qquad [29]$$

$$D_n = \frac{\int_0^{\ell} (\dot{y}_o Y_n + r^2 \dot{\psi}_o \Psi_n)\, dx}{\int_0^{\ell} (Y_n^2 + r^2 \Psi_n^2)\, dx} \qquad [30]$$

Substitution of [27] and [28] in [16] results in

$$T_n(0) + \sum_{i=1}^{4} G_{in} f_i(0) = C_n \qquad [31]$$
$$\dot{T}_n(0) + \sum_{i=1}^{4} G_{in} \dot{f}_i(0) = D_n$$

From [26] we have

$$T_n(0) = A_n \qquad [32]$$
$$\dot{T}_n(0) = \omega_n B_n$$

Thus

$$A_n = C_n - \sum_{i=1}^{4} G_{in} f_i(0) \qquad [33]$$
$$B_n = \frac{1}{\omega_n}\left[D_n - \sum_{i=1}^{4} G_{in} \dot{f}_i(0)\right]$$

This completes the formal solution of the problem.

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